Int J Theor Phys (2008) 47: 3321–3334
DOI 10.1007/s10773-008-9767-1

Quantization of Two Classical Models by Means of the BRST Quantization Method

Paul Bracken

Received: 21 March 2008 / Accepted: 28 May 2008 / Published online: 6 June 2008
© Springer Science+Business Media, LLC 2008

Abstract An elementary gauge-non-invariant model and the bosonized form of the chiral Schwinger model are introduced as classical theories. The constraint structure is then investigated. It is shown that by introducing a new field, these models can be made gauge-invariant. The BRST form of quantization is reviewed and applied to each of these models in turn such that gauge-invariance is not broken. Some consequences of this form of quantization are discussed.

Keywords Quantization · Constraints · Gauge-invariant · BRST approach · Poisson brackets · Ghosts

1 Introduction

The quantization of classically formulated models and the investigation of their properties is of particular importance in the study of quantum field theory, especially for the case in which the theory possesses constraints. It is often the case that second class constraints appear during the development of a theory and its quantization. It is of importance to handle any constraints there may be properly when they arise. This is especially true in the formulation of a gauge theory, and likely will be of similar importance in the quantization of gravity.

The standard method which is originally due to Dirac [2] for the quantization of these theories requires that Dirac brackets be calculated with respect to the field variables and the constraints of the theory. At the end, the Dirac brackets are replaced by quantum commutators. Another way of approaching the problem, which is very different from Dirac's method, is more recent. It is applicable to gauge theories and of great theoretical interest in itself. It is usually called the Batalin-Fradkin-Vilkovisky (BFT) or BRST quantization scheme [4, 5].

Recently, the bosonized version of the Jackiw-Rajaraman model has been studied by means of Dirac's method of quantization [1]. It is the intention here to continue this work by carrying out a quantization of this model by using the BRST approach. It is hoped that

P. Bracken (✉)
Department of Mathematics, University of Texas, Edinburg, TX 78541-2999, USA
e-mail: bracken@panam.edu

in seeing it applied to these kinds of models, its application to more realistic models will be made more amenable. An introduction to the general procedure for this method of quantization will be reviewed. Two different models will be introduced and then quantized in this scheme. It is the intention here to study a single, self-dual model of chiral bosons under this form of quantization [7, 8]. It will then be applied to generate a quantization of the Jackiw-Rajaraman model [6], which is a bosonized form of the chiral Schwinger model. Each of these models is made gauge-invariant by the introduction of an additional new field.

The advantages for using such a method are grounded in physics and we will quickly mention why. The standard problem with the usual Hamiltonian formulation of a gauge-invariant theory when gauge fixing conditions are required is that gauge invariance is broken when a specific gauge is chosen. To fix a gauge essentially converts a set of first-class constraints into a set of second-class constraints, which implies that gauge invariance of the theory has been lost. The importance of the BRST approach to the quantization of a theory is that the procedure preserves gauge invariance while producing a quantization scheme. The theory is constructed such that it possesses a generalized gauge invariance which is simply referred to as BRST symmetry. To carry out this form of quantization, the Hilbert space of the gauge invariant theory must be expanded in such a way that the idea of a gauge transformation is replaced by the notion of a BRST transformation. This new transformation can be said to have the property that it shifts or transforms operators which have different statistics among themselves. In this sense, a gauge invariant theory is embedded inside a larger theory, namely a theory which has this type of generalized symmetry. At the end, the states of the BRST theory must be projected out to produce the actual physical states of the theory. In fact, projecting any state onto the sector of BRST and anti-BRST invariant states produces a theory which is isomorphic to the original gauge invariant theory.

2 Introduction to the BRST Quantization Procedure

To make this self-contained, let us sketch this alternate procedure to Dirac's method [3, 5]. Let the phase space variables be denoted by x_A with $A = 1, \dots, n$, and constraint functions $\phi_\alpha(x_A)$, $\alpha = 1, \dots, m$ with associated Lagrange multipliers λ^α, which are real. In the case in which the constraints are all first class, or have been made so, the brackets used may be taken to be Poisson brackets. The constraints and Hamiltonian H satisfy

$$\{\phi_\alpha, \phi_\beta\} = C^\gamma_{\alpha\beta}\phi_\gamma, \qquad \{H, \phi_\alpha\} = V^\beta_\alpha \phi_\beta. \tag{2.1}$$

In the BFV formalism, the original space of phase space variables x_A and Lagrange multipliers λ^α is extended to a larger phase space. To give the Lagrange multipliers λ^α the status of dynamical degrees of freedom, introduce π_α conjugate to λ^α with the same Grassman parity as λ^α and ϕ_α, such that $\pi^*_\alpha = \pi_\alpha$. The nonvanishing fundamental brackets involving these conjugate degrees of freedom λ^α and π_α are

$$\{\pi_\alpha, \lambda^\beta\} = -\delta^\beta_\alpha, \qquad \{\lambda^\alpha, \pi_\beta\} = (-1)^{\epsilon_\alpha}\delta^\alpha_\beta. \tag{2.2}$$

The introduction of these new degrees of freedom leads to new first class constraints

$$\pi_\alpha \approx 0.$$

These are first class since π_α have vanishing brackets, both with π_α and ϕ_α. The $2m$ constraint functions $(\phi_\alpha, \pi_\alpha)$ will be represented by G_a, for $a = 1, \dots, 2m$.

Next new degrees of freedom are introduced in order to compensate for the use of degrees of freedom among the set $(x_A, \lambda^\alpha, \pi_\alpha)$, which are not all independent. For each first class constraint G_a, a pair of conjugate ghosts (η^a, P_a) are introduced, which are of Grassmann parity opposite G_a. The only nonvanishing fundamental brackets involving the BFV ghosts are

$$\{P_a, \eta^b\} = -\delta_a^b, \qquad \{\eta^a, P_b\} = -(-1)^{\epsilon_a}\delta_b^a. \tag{2.3}$$

These BFV ghosts have the following properties under complex conjugation

$$(\eta^a)^* = \eta^a, \qquad (P_a)^* = -(-1)^{\epsilon_a} P_a. \tag{2.4}$$

The BFV extended phase space is then the space which is parametrized locally by the coordinates $\{x_A, \lambda^\alpha, \pi_\alpha, \eta^a, P_a\}$, with a geometry specified by the fundamental brackets (2.3) and (2.4). The system described by this extended phase space is also subject to the first class constraints.

One of the consequences of the BFV extension of phase space is the following important fact. There always exists an operator referred to as the BRST charge Q which is determined up to canonical transformations by the following five properties. (i) $Q^* = Q$, (ii) it has Grassmann parity $+1$ (iii) Q has ghost number $+1$ (iv) $\frac{\partial}{\partial \eta^a} Q = G_a$ (v) $\{Q, Q\} = 0$. These properties actually dictate the form of Q. In fact, for abelian theories, the generator reduces to the first term

$$Q = \eta^a G_a. \tag{2.5}$$

Therefore, corresponding to the local gauge transformations generated by the first-class constraints G_a, there are global BRST transformations in extended phase space which are generated by the BRST charge.

This should be apparent since the $\eta^a G_a$ term can be thought of as the generator $\zeta^a G_a$ of local gauge transformations with the parameter ghosts. In the case of a closed algebra of constraints, the BRST transformations can be written as

$$\delta_B x_A = \{x_A, G_a\}\eta^a, \qquad \delta_B \lambda^\alpha = (-1)^{\epsilon_\alpha}\eta^{\alpha(1)}, \qquad \delta_B \pi^\alpha = 0. \tag{2.6}$$

The existence and expression of the BRST charge are independent of any specific Hamiltonian and gauge fixing conditions. In the end, the local gauge invariance of the original system associated to the constraints ϕ_α is traded for a global BRST symmetry generated by Q. One of the goals of the paper is to see how the formalism works. It will be applied to two models, the gauge non-invariant Srivastava model and the bosonized Schwinger model.

3 The Gauge-Non-Invariant Srivastava Model

In the first instance, we consider the gauge-non-invariant Srivastava model for single self-dual chiral bosons. It is described by the Lagrangian density

$$\mathcal{L}^N = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\varphi'^2 + \lambda(\dot{\varphi} - \varphi'). \tag{3.1}$$

The Lorentz metric $g^{\mu\nu} = (1, -1)$ where $\mu, \nu = 0, 1$ applies and the derivatives in $\mathcal{L}^N$ correspond to $\partial_0 \varphi = \dot{\varphi}$ and $\partial_1 \varphi = \varphi'$. From $\mathcal{L}^N$, the associated momenta are calculated

$$\pi_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \dot{\varphi} + \lambda, \qquad \pi_\lambda = \frac{\partial \mathcal{L}^N}{\partial \dot{\lambda}} = 0. \tag{3.2}$$

Therefore, the model has the primary constraint

$$\Omega_1 = \pi_\lambda \approx 0. \tag{3.3}$$

Since $\dot{\varphi} = \pi_\varphi - \lambda$ the Hamiltonian density corresponding to $\mathcal{L}^N$ is found to be

$$\begin{aligned}
\mathcal{H}^N &= \pi_\varphi \dot{\varphi} + \pi_\lambda \dot{\lambda} - \mathcal{L}^N = \pi_\varphi(\pi_\varphi - \lambda) - \frac{1}{2}(\pi_\varphi - \lambda)^2 + \frac{1}{2}\varphi'^2 - \lambda(\pi_\varphi - \lambda - \varphi') \\
&= \pi_\varphi^2 - \lambda\pi_\varphi - \frac{1}{2}\pi_\varphi^2 + \lambda\pi_\varphi - \frac{1}{2}\lambda^2 + \frac{1}{2}\varphi'^2 + \lambda\pi_\varphi + \lambda^2 + \lambda\varphi' \\
&= \frac{1}{2}(\pi_\varphi - \lambda)^2 + \frac{1}{2}\varphi'^2 + \lambda\varphi'.
\end{aligned} \tag{3.4}$$

The total Hamiltonian density corresponding to $\mathcal{L}^N$ with w as a Lagrange multiplier field is given by

$$\mathcal{H}_T^N = \frac{1}{2}(\pi_\varphi - \lambda)^2 + \frac{1}{2}\varphi'^2 + \lambda\varphi' + w\pi_\lambda. \tag{3.5}$$

To determine the Poisson bracket of two quantities $A(x)$ and $B(y)$, the following is calculated

$$\{A(x), B(y)\} = \int dz \sum_\alpha \frac{\partial A(x)}{\partial q_\alpha(z)} \frac{\partial B(y)}{\partial p_\alpha(z)} - \frac{\partial A(x)}{\partial p_\alpha(z)} \frac{\partial B(y)}{\partial q_\alpha(z)}. \tag{3.6}$$

Requiring that π_λ be preserved in time, the Poisson bracket with H^N must vanish, thus

$$\dot{\pi}_\lambda = \{\pi_\lambda, H^N\} = \pi_\varphi(x) - \lambda(x) - \varphi'(x) \approx 0. \tag{3.7}$$

This leads to no further constraints, therefore, the following pairs of constraints for the model have been found,

$$\Omega_1 = \pi_\lambda \approx 0, \qquad \Omega_2 = \pi_\varphi - \lambda - \varphi' \approx 0. \tag{3.8}$$

These equations express weak equality in the sense of Dirac. The momenta π_φ, π_λ are canonically conjugate, respectively, to the fundamental fields φ and λ in the Lagrangian, which we group together (φ, π_φ), (λ, π_λ). Moreover, Ω_1 is a primary constraint and Ω_2 is a secondary constraint.

Thus, the model is not gauge invariant. It is remarkable that gauge invariance can be restored by expanding the model into a new one by introducing a new field referred to as ϑ [10]. This quantum field will be introduced by transforming, or redefining the existing fields φ and λ in the original Lagrangian density $\mathcal{L}^N$ as follows

$$\varphi \to \varphi - \vartheta, \qquad \lambda \to \lambda + \dot{\vartheta}. \tag{3.9}$$

The Lagrangian density becomes

$$\begin{aligned}
\mathcal{L}^I &= \frac{1}{2}(\dot{\varphi} - \dot{\vartheta})^2 - \frac{1}{2}(\varphi' - \vartheta')^2 + (\lambda + \dot{\vartheta})(\dot{\varphi} - \dot{\vartheta} - \varphi' + \vartheta') \\
&= \mathcal{L}^N - \frac{1}{2}\dot{\vartheta}^2 - \frac{1}{2}\vartheta'^2 + \varphi'\vartheta' + \dot{\vartheta}\vartheta' - \dot{\vartheta}\varphi' - \lambda(\dot{\vartheta} - \vartheta') \\
&= \mathcal{L}^N + \mathcal{L}^{WZ}.
\end{aligned} \tag{3.10}$$

In (3.10), $\mathcal{L}^{WZ}$ is referred to as a Wess-Zumino term corresponding to $\mathcal{L}^N$

$$\mathcal{L}^{WZ} = -\frac{1}{2}\dot{\vartheta}^2 - \frac{1}{2}\vartheta'^2 + \varphi'\vartheta' + \dot{\vartheta}\vartheta' - \dot{\vartheta}\varphi' - \lambda(\dot{\vartheta} - \vartheta'). \tag{3.11}$$

The momenta are determined by differentiating $\mathcal{L}^I$,

$$\pi_\varphi = \frac{\partial \mathcal{L}^I}{\partial \dot{\varphi}} = \dot{\varphi} + \lambda, \qquad \pi_\vartheta = \frac{\partial \mathcal{L}^I}{\partial \dot{\vartheta}} = -\dot{\vartheta} + \vartheta' - \varphi' - \lambda, \qquad \pi_\lambda = \frac{\partial \mathcal{L}^I}{\partial \dot{\lambda}} = 0. \tag{3.12}$$

These serve to determine $\dot{\varphi}$ and $\dot{\vartheta}$ in terms of π_φ and π_ϑ as follows

$$\dot{\varphi} = \pi_\varphi - \lambda, \qquad \dot{\vartheta} = -\pi_\vartheta + \vartheta' - \varphi' - \lambda. \tag{3.13}$$

From these, the Hamiltonian density which corresponds to $\mathcal{L}^I$ is then given by

$$\mathcal{H}^I = \pi_\varphi \dot{\varphi} + \pi_\vartheta \dot{\vartheta} + \pi_\lambda \dot{\lambda} - \mathcal{L}^I. \tag{3.14}$$

Substituting $\mathcal{L}^I$, replacing $\dot{\varphi}$ and $\dot{\vartheta}$ in (3.15) and simplifying, the total Hamiltonian density including a Lagrange multiplier field u becomes

$$\mathcal{H}_T^I = \frac{1}{2}\pi_\varphi^2 - \frac{1}{2}\pi_\vartheta^2 + \pi_\vartheta\vartheta' - \pi_\vartheta\varphi' - \lambda\pi_\varphi - \lambda\pi_\vartheta + u\pi_\lambda. \tag{3.15}$$

Thus including the multiplier u as a dynamical variable, we summarize the relevant fields (φ, π_φ), (λ, π_λ), $(\vartheta, \pi_\vartheta)$ and (u, p_u). The Hamiltonian is obtained by integrating $\mathcal{H}_T^I$ with respect to the x variable

$$H_T^I = \int \mathcal{H}_T^I \, dx. \tag{3.16}$$

From this, Hamilton's equations can be determined to be

$$\begin{aligned}
\dot{\varphi} &= \frac{\partial H_T^I}{\partial \pi_\varphi} = \pi_\varphi - \lambda, \qquad -\dot{\pi}_\varphi = \frac{\partial H_T^I}{\partial \varphi} = \pi_\vartheta', \\
\dot{\lambda} &= \frac{\partial H_T^I}{\partial \pi_\lambda} = u, \qquad -\dot{\pi}_\lambda = \frac{\partial H_T^I}{\partial \lambda} = -\pi_\varphi - \pi_\vartheta, \\
\dot{\vartheta} &= \frac{\partial H_T^I}{\partial \pi_\vartheta} = -\pi_\vartheta + \vartheta' - \varphi' - \lambda, \qquad -\dot{\pi}_\vartheta = \frac{\partial H_T^I}{\partial \vartheta} = -\pi_\vartheta', \\
\dot{u} &= \frac{\partial H_T^I}{\partial p_u} = 0, \qquad -p_u = \frac{\partial H_T^I}{\partial u} = \pi_\lambda.
\end{aligned} \tag{3.17}$$

Now $\mathcal{L}^I$ is still seen to possess one primary constraint

$$\Psi_1 = \pi_\lambda \approx 0. \tag{3.18}$$

The requirement that this constraint be preserved in time implies that $\dot{\pi}_\lambda = 0$. It follows from the corresponding Hamilton's equations in (3.17) that

$$\pi_\varphi + \pi_\vartheta = 0, \tag{3.19}$$

is a secondary constraint. Differentiating the secondary constraint $\Psi_2 = \pi_\varphi + \pi_\vartheta$ and using Hamilton's equations gives

$$\dot{\pi}_\varphi + \dot{\pi}_\vartheta = -\dot{\pi}'_\vartheta + \dot{\pi}'_\vartheta = 0.$$

There are thus no further constraints here. The matrix of Poisson brackets of the constraints Ψ_i is a 2×2 null matrix. This implies that the constraints Ψ_i are first-class and that the theory $\mathcal{L}^{\mathcal{I}}$ is a gauge invariant theory. Moreover, $\mathcal{L}^I$ is seen to be invariant under the time-dependent chiral gauge-transformations, which are given by

$$\delta\varphi = \delta\vartheta = \pm r(x,t), \qquad \delta\lambda = \mp\dot{r}(x,t), \qquad \delta\pi_\varphi = \delta\pi_\vartheta = \delta\pi_\lambda = 0. \tag{3.20}$$

Here $r(x,t)$ is an arbitrary function of the coordinates.

4 BRST Formulation of the Gauge-Invariant Model

In contrast to Dirac's approach, we would like to rewrite the gauge-invariant theory described by $\mathcal{L}^I$ in the form of a quantum system which has this generalized gauge invariance or BRST symmetry. To do so, the Hilbert space of the theory must be enlarged beyond that of the gauge invariant theory. To do so, Grassmann variables $\{\eta^a\}$ are introduced. Given the constraints G_a, we then calculate the transformations of the basic fields using (2.6). Since η^3 repeats in the results, to simplify let us introduce the anti-commuting variables c and its conjugate $\bar{c}$ such that $\eta^2 = \eta^3 = -c$ and $\eta^1 = \eta^4 = \dot{c}$. These are referred to as the Faddeev-Popov ghost and anti-ghost fields which will be Grassmann numbers at the classical level and operators in the quantized theory. As well, a commuting variable b called a Nakanishi-Lautrup field [9] is introduced including both sign cases such that the complete set of transformations are given by

$$\begin{aligned} &\delta_B\varphi = \pm c, \qquad \delta_B\lambda = \mp\dot{c}, \qquad \delta_B\vartheta = \pm c, \qquad \delta_B\pi = 0, \\ &\delta_B p_\lambda = 0, \qquad \delta_B\pi_\vartheta = 0, \qquad \delta_B c = 0, \qquad \delta_B\bar{c} = b, \qquad \delta_B b = 0. \end{aligned} \tag{4.1}$$

The operator δ_B has the property $\delta_B^2 = 0$. These rules will be used to modify the Lagrangian density so it has this BRST symmetry. Performing gauge-fixing in the BRST formalism implies adding to the first-order Lagrangian density $\mathcal{L}^I$ a BRST-invariant gauge-fixing term. Write the quantum Lagrangian density by first working out

$$\pi_\varphi\dot{\varphi} + \pi_\vartheta\dot{\vartheta} + p_u\dot{u} - \mathcal{H}^I = \pi_\varphi\dot{\varphi} + \pi_\vartheta\dot{\vartheta} + p_u\dot{u} - \frac{1}{2}\pi_\varphi^2 + \frac{1}{2}\pi_\vartheta^2 + \pi_\vartheta(\varphi' - \vartheta' + \lambda) + \pi_\varphi\lambda. \tag{4.2}$$

To obtain the modified or gauge fixed Lagrangian $\mathcal{L}_{BRST}$, a BRST-invariant term is added to (4.2),

$$\begin{aligned} \mathcal{L}_{BRST} = {} & \pi_\varphi\dot{\varphi} + \pi_\vartheta\dot{\vartheta} + p_u\dot{u} - \frac{1}{2}\pi_\varphi^2 + \frac{1}{2}\pi_\vartheta^2 + \pi_\vartheta(\varphi' - \vartheta' + \lambda) \\ & + \pi_\varphi\lambda + \delta_B\left\{\bar{c}\left(\dot{\lambda} - \varphi - \vartheta + \frac{1}{2}b\right)\right\}. \end{aligned} \tag{4.3}$$

Using the transformations (4.1), the last term in (4.3) becomes after an integration by parts

$$\delta_B\left\{\bar{c}\left(\dot{\lambda} - \varphi - \vartheta + \frac{1}{2}b\right)\right\} = b\left(\dot{\lambda} - \varphi - \vartheta + \frac{1}{2}b\right) - \bar{c}(\ddot{c} + 2c)$$

$$= \frac{1}{2}b^2 + b(\dot{\lambda} - \varphi - \vartheta) + \dot{\bar{c}}\dot{c} - 2\bar{c}c.$$

Hence, $\mathcal{L}_{BRST}$ is given by

$$\mathcal{L}_{BRST} = \pi\dot{\varphi} + \pi_\vartheta\dot{\vartheta} + p_u\dot{u} - \frac{1}{2}\pi_\varphi^2 + \frac{1}{2}\pi_\vartheta^2 + \pi_\vartheta(\varphi' - \vartheta' + \lambda) + \pi_\varphi\lambda + \frac{1}{2}b^2 + b(\dot{\lambda} - \varphi - \vartheta) + \dot{\bar{c}}\dot{c} - 2\bar{c}c. \tag{4.4}$$

The Euler-Lagrange equation for b arises from (4.4) as

$$b + \dot{\lambda} - \varphi - \vartheta = 0. \tag{4.5}$$

Using transformation rules (4.1), since $\delta_B b = 0$, from (4.5) it follows that

$$\ddot{c} + 2c = 0. \tag{4.6}$$

This operator is an oscillator. The bosonic momenta can be defined so that $p_\lambda = b$, the fermionic momenta are defined as

$$\pi_c = \dot{\bar{c}}, \qquad \pi_{\bar{c}} = \dot{c}. \tag{4.7}$$

This implies that the variable which is canonically conjugate to c is $\dot{\bar{c}}$ and the variable conjugate to $\bar{c}$ is $\dot{c}$.

The quantum Hamiltonian density is hermitean and has the form

$$\begin{aligned}\mathcal{H}_{BRST} &= \pi_\varphi\dot{\varphi} + \pi_\vartheta\dot{\vartheta} + \pi_\lambda\dot{\lambda} + p_u\dot{u} + \pi_c\dot{c} + \dot{\bar{c}}\pi_{\bar{c}} - \mathcal{L}_{BRST} \\ &= \frac{1}{2}\pi_\varphi^2 - \frac{1}{2}\pi_\vartheta^2 - \pi_\vartheta(\varphi' - \vartheta' + \lambda) - \pi_\varphi\lambda - \frac{1}{2}p_\lambda^2 + p_\lambda(\varphi + \vartheta) + \pi_c\pi_{\bar{c}} + 2\bar{c}c.\end{aligned} \tag{4.8}$$

The fermionic variables must satisfy

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\dot{\bar{c}}, \dot{c}\} = 0, \tag{4.9}$$

and moreover,

$$\{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\}. \tag{4.10}$$

The minus sign in this is nontrivial and implies the existence of states with negative norm in the space of states of the theory. To specify (4.10) accurately, based on (4.8), we calculate by means of the Heisenberg equation

$$i\dot{c} = [c, \mathcal{H}_{BRST}] = c\pi_c\pi_{\bar{c}} + \pi_c c\pi_{\bar{c}} = \{c, \pi_c\}\pi_{\bar{c}} = \{c, \dot{\bar{c}}\}\dot{c}.$$

Therefore, it is the case that

$$\{c, \dot{\bar{c}}\} = i. \tag{4.11}$$

The final thing to specify is the BRST charge. It can be written down using (2.5)

$$Q_B = \int (i(\pi_\varphi + \pi_\vartheta)c - i\pi_\lambda\dot{c})\, dx. \tag{4.12}$$

It follows from (4.9) and (4.10) that this operator is nilpotent, and it mixes operators with different statistics.

Given that (4.6) holds for c, a solution can be written down which expresses the Heisenberg operator $c(t)$ and the corresponding operator $\bar{c}(t)$ as follows

$$c(t) = e^{i\sqrt{2}t}\alpha + e^{-i\sqrt{2}t}\beta, \qquad \bar{c}(t) = e^{-i\sqrt{2}t}\alpha^\dagger + e^{i\sqrt{2}t}\beta^\dagger. \tag{4.13}$$

When $t=0$, (4.13) implies that the operators on the right satisfy

$$\begin{aligned} &c = c(0) = \alpha + \beta, \qquad \bar{c} = \bar{c}(0) = \alpha^\dagger + \beta^\dagger, \\ &\dot{c} = \dot{c}(0) = i\sqrt{2}(\alpha - \beta), \qquad \dot{\bar{c}} = \dot{\bar{c}} = -i\sqrt{2}(\alpha^\dagger - \beta^\dagger). \end{aligned} \tag{4.14}$$

Equations (4.9) and (4.10) imply that these operators satisfy the system

$$\alpha^2 + \{\alpha,\beta\} + \beta^2 = \alpha^{\dagger 2} + \{\alpha^\dagger,\beta^\dagger\} + \beta^{\dagger 2} = 0, \tag{4.15}$$

$$\begin{aligned} &\{\alpha,\alpha^\dagger\} + \{\alpha^\dagger,\beta\} + \{\alpha,\beta^\dagger\} + \{\beta,\beta^\dagger\} = 0, \\ &\{\alpha,\alpha^\dagger\} + \{\alpha^\dagger,\beta\} - \{\alpha,\beta^\dagger\} - \{\beta,\beta^\dagger\} = -\frac{1}{\sqrt{2}}, \\ &\{\alpha,\alpha^\dagger\} + \{\alpha,\beta^\dagger\} - \{\alpha^\dagger,\beta\} - \{\beta,\beta^\dagger\} = -\frac{1}{\sqrt{2}}, \\ &\{\alpha,\alpha^\dagger\} - \{\alpha,\beta^\dagger\} - \{\alpha^\dagger,\beta\} + \{\beta,\beta^\dagger\} = 0. \end{aligned} \tag{4.16}$$

If (4.15), (4.16) is regarded as a system, at least one solution can be found. A solution to (4.15) and (4.16) is given by

$$\begin{aligned} &\alpha^2 = \beta^2 = \alpha^{\dagger 2} = \beta^{\dagger 2} = \{\alpha,\beta\} = \{\alpha^\dagger,\beta\} = \{\alpha,\beta^\dagger\} = \{\alpha^\dagger,\beta^\dagger\} = 0, \\ &\{\alpha^\dagger,\alpha\} = -\frac{1}{2\sqrt{2}}, \qquad \{\beta^\dagger,\beta\} = \frac{1}{2\sqrt{2}}. \end{aligned} \tag{4.17}$$

Suppose $|0\rangle$ denotes the fermionic vacuum such that

$$\alpha|0\rangle = \beta|0\rangle = 0, \tag{4.18}$$

and is defined to have norm one. Then using (4.17(i)), it follows that

$$\{\alpha,\alpha^\dagger\}|0\rangle = \alpha\alpha^\dagger|0\rangle = -\frac{1}{2\sqrt{2}}|0\rangle. \tag{4.19}$$

This implies that

$$\langle 0|\alpha\alpha^\dagger|0\rangle = -\frac{1}{2\sqrt{2}}. \tag{4.20}$$

Similarly,

$$\{\beta,\beta^\dagger\}|0\rangle = \beta\beta^\dagger|0\rangle = \frac{1}{2\sqrt{2}}|0\rangle.$$

Thus it follows that

$$\langle 0|\beta\beta^\dagger|0\rangle = \frac{1}{2\sqrt{2}}. \tag{4.21}$$

These results imply that the state $|0\rangle$ satisfy

$$\alpha^{\dagger}|0\rangle \neq 0, \qquad \beta^{\dagger}|0\rangle \neq 0. \tag{4.22}$$

The anti-BRST transformations corresponding to (4.1) can also be written down, as well as the anti-BRST charge $\bar{Q}_B$ corresponding to (4.12).

5 Bosonized Schwinger Model

The Lagrangian density for the chiral Schwinger model is given explicitly in the form

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}[i\ \partial\!\!\!/ + e\sqrt{\pi}\ A\!\!\!/(1+i\gamma_5)]\psi. \tag{5.1}$$

It has been shown that an auxiliary scalar field can be introduced which connects (5.1) to the bosonized version of the model such that the Lagrangian in terms of the scalar field is

$$\mathcal{L}_S(\varphi, A) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}(\partial^{\mu}\varphi)(\partial_{\mu}\varphi) + e(g^{\mu\nu} - \epsilon^{\mu\nu})A_{\nu}\partial_{\mu}\varphi + \frac{1}{2}ae^2 A^{\mu}A_{\mu}. \tag{5.2}$$

The field equations which are associated to (5.2) are given by

$$\begin{aligned} \Box\varphi + e(g^{\mu\nu} - \epsilon^{\mu\nu})A_{\nu} &= 0, \\ \partial_{\mu}F^{\mu\nu} + e(g^{\mu\alpha} - \epsilon^{\nu\alpha})\partial_{\alpha}\varphi + ae^2 A^{\nu} &= 0. \end{aligned} \tag{5.3}$$

The bosonized version for the case in which $a = 1$ will be considered here. Thus the Lagrangian density with $e = 1$ can be expanded out as

$$\mathcal{L}_S = \frac{1}{2}(\dot{\varphi}^2 - \varphi^{'2}) + (\dot{\varphi} + \varphi')(A_0 - A_1) + \frac{1}{2}(\dot{A}_1 - A_0^{'})^2 + \frac{1}{2}(A_0^2 - A_1^2). \tag{5.4}$$

The first term corresponds to a massless boson, the second represents the chiral coupling of φ to the electromagnetic field A_{μ} and the third term is the kinetic energy of the electromagnetic field, and the last term can be associated with the vector particle mass.

The canonical momenta are determined from (5.4) to be

$$\pi_0 = \frac{\partial L_S}{\partial \dot{A}_0} = 0, \qquad \pi_1 = \frac{\partial L_S}{\partial \dot{A}_1} = \dot{A}_1 - A_0', \qquad \pi = \frac{\partial L_S}{\partial \dot{\varphi}} = \dot{\varphi} + A_0 - A_1. \tag{5.5}$$

The Hamiltonian density can be determined from these momenta

$$\mathcal{H} = \pi\dot{\varphi} + \pi_1\dot{A}_1 + \pi_0\dot{A}_0 - \mathcal{L}_S. \tag{5.6}$$

Using (5.5), $\mathcal{H}_S$ is determined to be

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}\pi_1^2 + \frac{1}{2}\varphi^{'2} + \pi_1 A_0' + (\pi + \varphi' + A_1)(A_1 - A_0). \tag{5.7}$$

The Hamiltonian is the integral of $\mathcal{H}$ with respect to x,

$$H_S = \int dx\, \mathcal{H}.$$

The canonically conjugate pairs can then be summarized in the form (φ, π), (A_0, π_0) and (A_1, π_1). The Lagrangian density (5.4) possesses the following four second-class constraints

$$\begin{aligned} \Omega_1 &= \pi_0 \approx 0, & \Omega_2 &= \pi_1' + \varphi' + \pi + A_1 \approx 0, \\ \Omega_3 &= \pi_1 \approx 0, & \Omega_4 &= -\pi - \varphi' - 2A_1 + A_0 \approx 0. \end{aligned} \tag{5.8}$$

The constraint Ω_1 is a primary constraint and Ω_2, Ω_3 and Ω_4 are secondary constraints. Based on the set of constraints $\{\Omega_a\}$, the matrix of Poisson brackets can be evaluated. It is found to be nonsingular and has an inverse. The Dirac brackets can therefore be evaluated, and Dirac's algorithm for quantizing this system can be applied to these results. This effectively removes second class constraints from the problem.

A gauge-invariant model can be developed which corresponds to the Lagrangian in (5.4). One way to do this is to introduce a new scalar field which corresponds to a Wess-Zumino term. To carry this out, the actual Hilbert space of the theory is expanded to include a new scalar field as in the previous model. The fields φ and A_μ in the original Lagrangian can be redefined using a new field ϑ under the following

$$\varphi \to \varphi - \vartheta, \qquad A^\mu \to A^\mu + \partial^\mu \vartheta. \tag{5.9}$$

Under such a replacement, the Lagrangian $\mathcal{L}_S$ is mapped into $\mathcal{L}_T$ which is given by

$$\begin{aligned} \mathcal{L}_T &= \frac{1}{2}(\dot\varphi^2 - \varphi'^2) + (\dot\varphi + \varphi')(A_0 - A_1) + \frac{1}{2}(\dot A_1 - A_0')^2 \\ &\quad + \frac{1}{2}(A_0^2 - A_1^2) + \varphi'\dot\vartheta - \dot\varphi\vartheta' + \dot\vartheta A_1 - \vartheta' A_0 \\ &= \mathcal{L}_S + \mathcal{L}_\vartheta. \end{aligned} \tag{5.10}$$

In fact $\mathcal{L}_T$ describes a gauge-invariant theory. It will then be of interest to quantize this theory according to the BRST formalism. The Euler-Lagrange equations which are obtained from $\mathcal{L}_T$ are given by

$$\begin{aligned} \ddot\varphi - \varphi'' &= \dot A_1 - A_0' - \dot A_0 + A_1', \\ \ddot A_1 - \dot A_0' &= \dot\vartheta - \dot\varphi - \varphi' - A_1, \\ \dot A_1' - A_0'' &= \vartheta' - A_0 - \dot\varphi - \varphi', \\ \dot A_1 - A_0' &= 0. \end{aligned} \tag{5.11}$$

The canonical momenta for the gauge-invariant theory are found to be

$$\begin{aligned} \pi_0 &= \frac{\partial L_T}{\partial \dot A_0} = 0, & \pi_\vartheta &= \frac{\partial L_T}{\partial \dot\vartheta} = A_1 + \varphi', \\ \pi_1 &= \frac{\partial L_T}{\partial \dot A_1} = \dot A_1 - A_0', & \pi &= \frac{\partial L_T}{\partial \dot\varphi} = \dot\varphi + A_0 - A_1 - \vartheta'. \end{aligned} \tag{5.12}$$

Therefore, the theory has two primary constraints, each independent of velocity terms

$$\Omega_1 = \pi_0 \approx 0, \qquad \Omega_2 = \pi_\vartheta - A_1 - \varphi' \approx 0. \tag{5.13}$$

Only two of the momenta π and π_1 in (5.12) involve time derivatives of the fields, and it is possible to solve for these derivatives as follows

$$\dot A_1 = \pi_1 + A_0', \qquad \dot\varphi = \pi - A_0 + A_1 + \vartheta'. \tag{5.14}$$

The canonical Hamiltonian density can be calculated from $\mathcal{L}_T$ using (5.14) to be

$$\mathcal{H}_T = \pi\dot{\varphi} + \pi_\vartheta\dot{\vartheta} + \pi_1\dot{A}_1 + \pi_0\dot{A}_0 - \mathcal{L}_T. \tag{5.15}$$

The primary constraints can be included in the canonical Hamiltonian density by making use of a pair of Lagrange multipliers λ_0 and λ_1 as follows

$$\begin{aligned}\mathcal{H}_E &= \mathcal{H}_T + \lambda_0\pi_0 + \lambda_1(\pi_0 - A_1 - \varphi')\\ &= \frac{1}{2}(\pi^2 + \pi_1^2) + \frac{1}{2}(\varphi'^2 + \vartheta'^2) + \pi_1 A_0' + (\pi + \varphi' + A_1 + \vartheta')(A_1 - A_0) + \vartheta' A_0\\ &\quad + \pi\vartheta' + \pi_0\lambda_0 + (\pi_\vartheta - A_1 - \varphi')\lambda_1. \end{aligned}\tag{5.16}$$

The total Hamiltonian is given by the integral of $\mathcal{H}_E$ over the spatial coordinate. From the total Hamiltonian, the entire set of Hamilton's equations can be worked out [1].

The two primary constraints are Ω_1 and Ω_2, and it is required that these constraints be preserved with respect to time. Demanding that the primary constraint Ω_1 be preserved in time, a secondary constraint is obtained. Since $\dot{\pi}$ is determined by one of Hamilton's equations for the model, it follows that

$$\{\Psi_1, H_E\} = \dot{\pi}_0 = \pi_1' + \pi + \varphi' + A_1 \approx 0.$$

This implies that there is a third constraint

$$\Omega_3 = \pi_1' + \pi + \varphi' + A_1 \approx 0. \tag{5.17}$$

Requiring that this constraint be preserved in time leads to a fourth constraint

$$\dot{\Omega}_3 = \dot{\pi}_1' + \dot{\pi} + \dot{\varphi}' + \dot{A}_1 = \pi_1 + \lambda_1'.$$

It suffices to make λ_1 independent of x. Requiring that Ω_1 and Ω_4 be preserved with respect to time leads to no further constraints on the theory.

Therefore, the theory is seen to possess four constraints which are summarized here

$$\begin{aligned}&\Omega_1 = \pi_0 \approx 0, \qquad \Omega_2 = \pi_\vartheta - A_1 - \varphi' \approx 0,\\ &\Omega_3 = \pi_1' + \pi + \varphi' + A_1 \approx 0, \quad \Omega_4 = \pi_1 \approx 0.\end{aligned}\tag{5.18}$$

It is important to summarize the field variable pairs for the system

$$(\varphi, \pi), \qquad (A_0, \pi_0), \qquad (A_1, \pi_1), \qquad (\vartheta, \pi_\vartheta), \qquad (\lambda_0, p_{\lambda_0}), \qquad (\lambda_1, p_{\lambda_1}). \tag{5.19}$$

Moreover, a current can be defined which is conserved, and it is given by $J^\nu = \partial_\mu F^{\mu\nu}$. Using the equations of motion, we can calculate

$$\begin{aligned}-\partial_\nu(\partial_\mu F^{\mu\nu}) &= (g^{\nu\alpha} - \epsilon^{\nu\alpha})\partial_\nu\partial_\alpha\varphi + \partial_\nu A^\nu = \dot{A}_0 - A_1' + \ddot{\varphi} - \varphi''\\ &= \dot{A}_0 - A_1' + \dot{A}_1 - A_0' - \dot{A}_0 + A_1' = \dot{A}_1 - A_0' = 0.\end{aligned}$$

This implies that $\partial_\nu J^\nu = 0$, and therefore the gauge-invariant theory is nonanomalous. This theory can be quantized by using Dirac's procedure which requires the introduction of two gauge fixing constraints. The theory can also be quantized without breaking gauge invariance by means of the BRST method. This is the subject of the next section.

6 BRST Quantization

In order to write down the BRST quantization of the gauge-invariant theory, the total Hamiltonian density (5.16) is converted into the first-order Lagrangian density

$$\begin{aligned}\mathcal{L}_T &= \pi\dot{\varphi} + \pi_1\dot{A}_1 + \pi_0\dot{A}_0 + \pi_\vartheta\dot{\vartheta} + p_{\lambda_0}\dot{\lambda}_0 + p_{\lambda_1}\dot{\lambda}_1 - \mathcal{H}_E \\ &= \pi\dot{\varphi} + \pi_1\dot{A}_1 + \pi_0\dot{A}_0 + \pi_\vartheta\dot{\vartheta} + p_{\lambda_0}\dot{\lambda}_0 + p_{\lambda_1}\dot{\lambda}_1 \\ &\quad - \frac{1}{2}(\pi^2 + \pi_1^2) - \frac{1}{2}(\varphi^{'2} + \vartheta^{'2}) - \pi_1 A_0' \\ &\quad - (\pi + \varphi' + A_1)(A_1 - A_0) - \vartheta' A_1 - \pi\vartheta' - \pi_0\lambda_0 - (\pi_\vartheta - A_1 - \varphi')\lambda_1. \quad (6.1)\end{aligned}$$

Given the constraints obtained in Sect. 5, the BRST transformations for the basic variables can be calculated using (2.6). These are, $\delta_B\varphi = \{\varphi, G_a\}\eta^a = \eta^3$, $\delta_B\vartheta = \eta^2$, $\delta_B A_0 = -\dot{\eta}^3$, $\delta_B A_1 = -\eta^{3'}$, $\delta_B\pi = -\eta^2 + \eta^3$, $\delta_B\pi_\vartheta = 0$, $\delta_B\pi_0 = 0$, $\delta_B\pi_1 = 0$. The Hilbert space of the gauge-invariant model is expanded. The idea of gauge transformations, which shift operators by c-number functions, is replaced by BRST transformations which mix operators with Bose and Fermi statistics. It suffices to take $\eta^1 = -\dot{c}$, $\eta^2 = \eta^3 = \eta^4 = -c$. Again introducing a commuting Nakanishi-Lautrup field, b, the set of transformations can be summarized as,

$$\begin{aligned}&\delta_B\varphi = \mp c = \delta_B\vartheta, \quad \delta_B A_0 = \pm\dot{c}, \quad \delta_B A_1 = \pm c', \quad \delta_B\pi = 0, \quad \delta_B\pi_\vartheta = 0, \\ &\delta_B\pi_0 = 0, \quad \delta_B\pi_1 = 0, \quad \delta_B c = 0, \quad \delta_B\bar{c} = b, \quad \delta_B = 0,\end{aligned} \tag{6.2}$$

with the property that $\delta_B^2 = 0$. A BRST-invariant function of the dynamical variables is defined to be a function $f(\pi, \pi_0, \pi_1, \pi_\vartheta, p_\lambda, \pi_c, \pi_{\bar{c}}, \varphi, A_0, A_1, \vartheta, b, c, \bar{c})$ such that $\delta_B f = 0$.

To perform gauge fixing in the BRST formalism, begin by adding to the first order Lagrangian density (6.1) a trivial BRST-invariant function.

$$\begin{aligned}\mathcal{L}_{BRST} &= \mathcal{L}_T + \delta_B\left[\bar{c}\left(\dot{A}_0 + \frac{1}{2}b - \vartheta - \varphi + \pi\right)\right] \\ &= \mathcal{L}_T + b\left(\dot{A}_0 + \frac{1}{2}b - \vartheta - \varphi + \pi\right) + \dot{\bar{c}}\dot{c} - 2\bar{c}c.\end{aligned} \tag{6.3}$$

To work out the Euler-Lagrange equation for b classically amounts to differentiating $\mathcal{L}_{BRST}$ with respect to b and this gives

$$-b = \dot{A}_0 - \vartheta - \varphi + \pi.$$

From (6.2), there is the requirement that $\delta_B = 0$, so this equation implies that

$$-\delta_B b = \delta_B\dot{A}_0 - \delta_B\vartheta - \delta_B\varphi + \delta_B\pi.$$

This in turn implies that c is an oscillator

$$\ddot{c} + 2c = 0. \tag{6.4}$$

In introducing momenta for the variable c, care must be taken in defining those for the fermionic variables. Thus the bosonic momenta are defined in the usual way

$$\pi_0 = \frac{\partial}{\partial\dot{A}_0}\mathcal{L}_{BRST} = b. \tag{6.5}$$

For the fermionic momenta with directional derivatives, we set

$$\pi_c = \mathcal{L}_{BRST} \frac{\overleftarrow{\partial}}{\partial \dot{c}} = \dot{\bar{c}}, \qquad \pi_{\bar{c}} = \frac{\partial}{\partial \dot{\bar{c}}} \mathcal{L}_{BRST} = \dot{c}, \tag{6.6}$$

implying that the variable canonically conjugate to c is $\dot{\bar{c}}$, and the variable conjugate to $\bar{c}$ is $\dot{c}$. The Hamiltonian density $\mathcal{H}_{BRST}$ is found from the Lagrangian density in the usual way, and must be Hermitian

$$\begin{aligned}
\mathcal{H}_{BRST} &= \pi\dot{\varphi} + \pi_0 \dot{A}_0 + \pi_1 \dot{A}_1 + \pi_\vartheta \dot{\vartheta} + p_{\lambda_0}\dot{\lambda}_0 + p_{\lambda_1}\dot{\lambda}_1 + \pi_c \dot{c} + \dot{\bar{c}}\pi_{\bar{c}} - \mathcal{L}_{BRST} \\
&= \pi_c \dot{c} + \dot{\bar{c}}\pi_{\bar{c}} + \frac{1}{2}(\pi^2 + \pi_1^2 + \varphi'^2 + \vartheta'^2) + \pi_1 A_0' + (\pi + \varphi' + A_1)(A_1 - A_0) \\
&\quad + \vartheta' A_1 + \pi\vartheta' - \frac{1}{2}b^2 - b(\dot{A}_0 - \vartheta - \varphi + \pi) - \dot{\bar{c}}\dot{c} + 2\bar{c}c.
\end{aligned} \tag{6.7}$$

Using (6.5) we set $\pi_0 = b$ with $\pi_c = \dot{\bar{c}}$ and $\pi_{\bar{c}} = \dot{c}$ in (6.7) to obtain

$$\begin{aligned}
\mathcal{H}_{BRST} &= \frac{1}{2}(\pi^2 + \pi_1^2 + \varphi'^2 + \vartheta'^2) + \pi_1 A_0' + (\pi + \varphi' + A_1)(A_1 - A_0) + \vartheta' A_1 + \pi\vartheta' \\
&\quad - \frac{1}{2}\pi_0^2 - \pi_0(\dot{A}_0 - \vartheta - \varphi + \pi) + \pi_c \pi_{\bar{c}} + 2\bar{c}c.
\end{aligned} \tag{6.8}$$

The consistency of $\mathcal{H}_{BRST}$ in (6.8) can be checked by using it to work out Hamilton's equations for the fermionic variables. These are found to agree with (6.6). The fermionic variables are assumed to anticommute, so it must be that

$$\{\bar{c}, c\} = \{\pi_c, \pi_{\bar{c}}\} = 0,$$

Differentiating $\{\bar{c}, c\}$ with respect to t implies that $\{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\}$. As with the previous model, demanding that c satisfy the Heisenberg equation $[c, \mathcal{H}_{BRST}] = i\dot{c}$, the following results

$$[c, \mathcal{H}_{BRST}] = c\pi_c\pi_{\bar{c}} - \pi_c\pi_{\bar{c}}c = c\dot{\bar{c}}\dot{c} - \dot{\bar{c}}\dot{c}c = \{c, \dot{\bar{c}}\}\dot{c}. \tag{6.9}$$

This must match $i\dot{c}$, which implies that as before, $\{c, \dot{\bar{c}}\} = i$. These results can be summarized in the form

$$c^2 = \bar{c}^2 = \{\bar{c}, c\} = \{\dot{\bar{c}}, \dot{c}\} = 0, \qquad \{\dot{\bar{c}}, c\} = -\{\dot{c}, \bar{c}\} = i. \tag{6.10}$$

The minus sign here implies the existence of states with negative norm in the space of state vectors of the theory.

It remains to specify the BRST charge operator Q_B, which is the generator of BRST transformations. It is nilpotent and therefore satisfies $Q_B^2 = 0$, and mixes operators that satisfy Bose and Fermi statistics. According to (2.5), we would write

$$Q_B = \int [(\pi + \pi_\vartheta + \pi_1 + \pi_1')c + \pi_0 \dot{c}]\, dx. \tag{6.11}$$

The set of states which satisfy the constraints $\Psi_i \approx 0$ belongs to the dynamically stable subspace of states $|\Phi\rangle$ which satisfy $Q_B|\Phi\rangle = 0$, that is, it belongs to the set of BRST-invariant states.

To understand the condition needed for recovering the physical states, the operators c and $\bar{c}$ are rewritten in terms of fermionic annihilation and creation operators. To do so, consider (6.4). The solution of this equation gives the specific form of the Heisenberg operators $c(t)$ and $\bar{c}(t)$ as in the previous model in (4.13). Defining $|0\rangle$ to have norm one, as before

$$\langle 0|\alpha\alpha^{\dagger}|0\rangle = -\frac{1}{2\sqrt{2}}, \qquad \langle 0|\beta\beta^{\dagger}|0\rangle = \frac{1}{2\sqrt{2}}.$$

Therefore $\alpha^{\dagger}|0\rangle \neq 0$ and $\beta^{\dagger}|0\rangle \neq 0$. This theory is seen to possess negative norm states in the fermionic sector. The existence of these negative norm states as free states of the fermionic part of $\mathcal{H}_{BRST}$ is, however, irrelevant to the existence of physical states in the orthogonal subspace of the Hilbert space.

Also $\mathcal{H}_{BRST}$ and Q_B can be written in terms of α and β. As well, the anti-BRST transformations and charge can be written down based on (6.2) and (6.11). It is the case that $[Q, H_{BRST}] = 0$ and $[\bar{Q}_B, H] = 0$, and we further impose the dual condition that both Q_B and $\bar{Q}_B$ annihilate physical states, which implies

$$Q_B|\Psi\rangle = 0, \qquad \bar{Q}_B|\Psi\rangle = 0. \tag{6.12}$$

The states for which (5.18) hold strongly satisfy both of these conditions and are the only states satisfying both conditions since although

$$4(\alpha^{\dagger}\alpha + \beta^{\dagger}\beta) = -4(\alpha\alpha^{\dagger} + \beta\beta^{\dagger}),$$

there are no states of this operator with $\alpha^{\dagger}|0\rangle = 0$, and $\beta^{\dagger}|0\rangle = 0$. Hence there are no free eigenstates of the fermionic part of $\mathcal{H}_{BRST}$ that are annihilated by each of α, $\alpha^{\dagger}$, β and $\beta^{\dagger}$.

In fact, the states which vanish when the constraints (5.18) in operator form are applied to a state satisfy (6.12), and are the only states satisfying both these relations. This is due to the fact that in view of (6.10), c, $\dot{c}$ and $\bar{c}$, $\dot{\bar{c}}$ cannot be simultaneously be applied to a state $|\Psi\rangle$ to give zero.

References

1. Bracken, P.: A chiral schwinger model its constraint structure and applications to its quantization. Int. J. Theor. Phys. A **23**, 855 (2008)
2. Dirac, P.A.M.: Lectures on Quantum Mechanics. Dover, Mineola (2001)
3. Govaerts, J.: Hamiltonian Quantization and Constrained Dynamics. Leuven Notes in Mathematical and Theoretical Physics, vol. 4. Leuven University Press, Leuven (1991)
4. Henneaux, M.: Hamiltonian form of the path integral for theories with a gauge freedom. Phys. Rep. **126**, 1 (1985)
5. Henneaux, M., Teitelboim, C.: Quantization of Gauge Systems. Princeton University Press, Princeton (1992)
6. Jackiw, R., Rajamaran, R.: Phys. Rev. Lett. **54**, 1219 (1985)
7. Kulshreshtha, U., Kulshreshtha, D., Müller-Kirsten, H.J.W.: Z. Phys. C **60**, 427 (1993)
8. Kulshreshtha, U., Kulshreshtha, D., Müller-Kirsten, H.J.W.: Can. J. Phys. **72**, 639 (1994)
9. Weinberg, S.: The Quantum Theory of Fields, vol. 2. Cambridge University Press, Cambridge (1996)
10. Wess, J., Zumino, B.: Phys. Lett. B **37**, 95 (1971)